# SEMILATTICE STRUCTURES OF SPREADING MODELS

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Abstract. Given a Banach space $X$, denote by $SP_w(X)$ the set of equivalence classes of spreading models of $X$ generated by normalized weakly null sequences in $X$. It is known that $SP_w(X)$ is a semilattice, i.e., it is a partially ordered set in which every pair of elements has a least upper bound. We show that every countable semilattice that does not contain an infinite increasing sequence is order isomorphic to $SP_w(X)$ for some separable Banach space $X$.

Given a normalized basic sequence $(y_i)$ in a Banach space and $\varepsilon_n \searrow 0$, using Ramsey's Theorem, one can find a subsequence $(x_i)$ and a normalized basic sequence $(\tilde{x}_i)$ such that for all $n \in \mathbb{N}$ and $(a_i)_{i=1}^n \subseteq [-1, 1]$,

$$|\|\sum a_i x_{k_i}\| - \|\sum a_i \tilde{x}_i\|| < \varepsilon_n$$

for all $n \leq k_1 < \cdots < k_n$. The sequence $(\tilde{x}_i)$ is called a spreading model of $(x_i)$. It is well-known that if $(x_i)$ is in addition weakly null, then $(\tilde{x}_i)$ is 1-spreading and suppression 1-unconditional. See [3, 5] for more about spreading models. A spreading model $(\tilde{x}_i)$ is said to ($C$-) *dominate* another spreading model $(\tilde{y}_i)$ if there is a $C < \infty$ such that for all $(a_i) \subseteq \mathbb{R}$,

$$\|\sum a_i \tilde{y}_i\| \leq C\|\sum a_i \tilde{x}_i\|.$$

The spreading models $(\tilde{x}_i)$ and $(\tilde{y}_i)$ are said to be *equivalent* if they dominate each other. Let $[(\tilde{x}_i)]$ denote the class of all spreading models which are equivalent to $(\tilde{x}_i)$. Let $SP_w(X)$ denote the set of all $[(\tilde{x}_i)]$ generated by normalized weakly null sequences in $X$. If $[(\tilde{x}_i)]$, $[(\tilde{y}_i)] \in SP_w(X)$, we write $[(\tilde{x}_i)] \leq [(\tilde{y}_i)]$ if $(\tilde{y}_i)$ dominates $(\tilde{x}_i)$. $(SP_w(X), \leq)$ is a partially ordered set. The paper [2] initiated the study of the order structures of $SP_w(X)$. It was established that every countable subset of $(SP_w(X), \leq)$ admits an upper bound ([2, Proposition 3.2]). Moreover, from the proof of this result, it follows that every pair of elements in $(SP_w(X), \leq)$ has a least upper bound. In other words, $(SP_w(X), \leq)$ is a *semilattice*. In [6], it was shown that if $SP_w(X)$ is countable, then it cannot admit a strictly increasing infinite sequence $(\tilde{x}_i^1) < (\tilde{x}_i^2) < \cdots$. In [4], two methods of construction, utilizing Lorentz sequence spaces and Orlicz sequence spaces respectively, were used to produce Banach spaces $X$ so that $SP_w(X)$ has certain prescribed order

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structures. In the present paper, building on the techniques employed in [4, §2], we show that every countable semilattice that has no infinite increasing sequence is order isomorphic to $SP_w(X)$ for some Banach space $X$. This gives an affirmative answer to Problem 1.15 in [4]. (See, however, the remark at the end of the paper.)

## 1. A Representation Theorem for Semilattices

Any collection of subsets of a set $V$ that is closed under the taking of finite unions is a semilattice under the order of set inclusion. In this section, we show that any countable semilattice that does not admit an infinite increasing sequence may be represented in such a way using a countable set $V$. The result may be of independent interest.

**Theorem 1.** *Let $L$ be a countable semilattice with no infinite increasing sequences. Then there exist a countable set $V$ and an injective map $T : L \to 2^V \setminus \{\emptyset\}$ that preserves the semilattice structure of $L$, i.e., $T(x \vee y) = T(x) \cup T(y)$ for all $x, y \in L$.*

Suppose that $L$ is a semilattice that satisfies the hypothesis of Theorem 1. Note that every nonempty subset of $L$ has at least one maximal element; for otherwise, it will admit an infinite increasing sequence. Set $L_0 = L$. If $L_\alpha$ is defined for some countable ordinal $\alpha$ and $L_\alpha \neq \emptyset$, let $L_{\alpha+1} = L_\alpha \setminus \{\text{maximal elements in } L_\alpha\}$. If $\alpha$ is a countable limit ordinal such that $L_{\alpha'} \neq \emptyset$ for all $\alpha' < \alpha$, let $L_\alpha = \cap_{\alpha'<\alpha} L_{\alpha'}$. Since $(L_\alpha)$ is a strictly decreasing transfinite sequence of subsets of the countable set $L$, $L_\alpha = \emptyset$ for some countable ordinal $\alpha$. Let $\alpha_0$ be the smallest ordinal such that $L_{\alpha_0} = \emptyset$. Enumerate $L$ as a transfinite sequence $(e_\beta)_{\beta<\beta_0}$ so that if $e_{\beta_1} \in L_{\alpha_1} \setminus L_{\alpha_1+1}$ and $e_{\beta_2} \in L_{\alpha_2} \setminus L_{\alpha_2+1}$ for some $\alpha_1 < \alpha_2 < \alpha_0$, then $\beta_1 < \beta_2$. If $1 \leq \beta \leq \beta_0$, let $U_\beta = \{e_{\beta'} : \beta' < \beta\}$. Note that $L = U_{\beta_0}$.

**Lemma 2.** (a) *$e_\beta$ is a minimal element in $U_{\beta+1}$.*
(b) *If $e_\beta = e_{\beta_1} \vee e_{\beta_2}$ (least upper bound taken in $L$), then $\beta \leq \min\{\beta_1, \beta_2\}$.*
(c) *If $e_{\beta_1}, e_{\beta_2} \in U_\beta$, then $e_{\beta_1} \vee e_{\beta_2}$ belongs to $U_\beta$.*

*Proof.* (a) Suppose on the contrary that $e_\beta$ is not a minimal element in $U_{\beta+1}$. Then there exists $e_{\beta'} \in U_{\beta+1}$ with $e_{\beta'} < e_\beta$. It follows from the definition of $U_{\beta+1}$ that $\beta' < \beta$. If $e_\beta \in L_\alpha \setminus L_{\alpha+1}$ and $e_{\beta'} \in L_{\alpha'} \setminus L_{\alpha'+1}$, then $\alpha' \leq \alpha$ and hence $L_\alpha \subseteq L_{\alpha'}$. Since $e_\beta, e_{\beta'} \in L_{\alpha'}$ and $e_{\beta'} < e_\beta$, $e_{\beta'}$ is not maximal in $L_{\alpha'}$. Thus $e_{\beta'} \in L_{\alpha'+1}$, a contradiction.

(b) Suppose that $\beta_1 < \beta$. Then $e_{\beta_1} \in U_{\beta+1}$ and $e_{\beta_1} < e_\beta$, contrary to the minimality of $e_\beta$ in $U_{\beta+1}$. Similarly, $\beta_2 \geq \beta$.

(c) Follows immediately from (b). $\square$

If $1 \leq \beta < \omega_1$, write $\beta = \gamma + n$, where $\gamma$ is a limit ordinal, $n < \omega$, and let $V_\beta$ denote the ordinal interval $[0, \gamma + 2n)$. We define a family of maps $T_\beta : U_\beta \to 2^{V_\beta} \setminus \{\emptyset\}$, $1 \leq \beta \leq \beta_0$, inductively so that $T = T_{\beta_0}$ is the map

sought for in Theorem 1. Let $T_1 : U_1 = \{e_0\} \to 2^{V_1} \setminus \{\emptyset\}$ be defined by $T_1(e_0) = \{0,1\}$. If $T_\beta$ has been defined, $1 \leq \beta < \beta_0$, let

$$T_{\beta+1}(x) = \begin{cases} T_\beta(x) \cup \{\gamma+2n, \gamma+2n+1\} & \text{if} \quad x \in U_{\beta+1} \setminus \{e_\beta\}, \\ \bigcap\limits_{e_\beta < z \in U_\beta} T_\beta(z) \cup \{\gamma+2n+1\} & \text{if} \qquad x = e_\beta. \end{cases}$$

When $\beta \leq \beta_0$ is a limit ordinal and $e_{\beta'} \in U_\beta$, let $T_\beta(e_{\beta'}) = \cup_{\beta'<\xi<\beta} T_\xi(e_{\beta'})$. The next result, which shows the compatibility of the definitions of $T_\beta$ for different $\beta$'s, is the key to the subsequent arguments.

**Lemma 3.** *If $1 \leq \beta_1 < \beta_2 \leq \beta_0$ and $\beta_i = \gamma_i + n_i$, $i = 1, 2$, then*

$$T_{\beta_2}(e_{\beta_1}) = T_{\beta_1+1}(e_{\beta_1}) \cup [\gamma_1 + 2n_1 + 2, \gamma_2 + 2n_2).$$

*Proof.* If $\beta_2 = \beta_1 + 1$, the assertion holds clearly. Suppose that the assertion holds for some $\beta_2 > \beta_1$. By the definition of $T_{\beta_2+1}$,

$$\begin{aligned} T_{\beta_2+1}(e_{\beta_1}) &= T_{\beta_2}(e_{\beta_1}) \cup \{\gamma_2 + 2n_2, \gamma_2 + 2n_2 + 1\} \\ &= T_{\beta_1+1}(e_{\beta_1}) \cup [\gamma_1 + 2n_1 + 2, \gamma_2 + 2n_2) \\ &\qquad \cup \{\gamma_2 + 2n_2, \gamma_2 + 2n_2 + 1\} \\ &= T_{\beta_1+1}(e_{\beta_1}) \cup [\gamma_1 + 2n_1 + 2, \gamma_2 + 2n_2 + 2). \end{aligned}$$

Suppose that $\beta_2 \leq \beta_0$ is a limit ordinal and the assertion holds for all $\beta_1 < \xi < \beta_2$. For such $\xi$, let $\xi = \gamma_\xi + n_\xi$. By the inductive hypothesis,

$$T_\xi(e_{\beta_1}) = T_{\beta_1+1}(e_{\beta_1}) \cup [\gamma_1 + 2n_1 + 2, \gamma_\xi + 2n_\xi).$$

Since $\beta_2$ is a limit ordinal, we have

$$\begin{aligned} T_{\beta_2}(e_{\beta_1}) &= \cup_{\beta_1<\xi<\beta_2} T_\xi(e_{\beta_1}) \\ &= T_{\beta_1+1}(e_{\beta_1}) \cup [\gamma_1 + 2n_1 + 2, \beta_2) \\ &= T_{\beta_1+1}(e_{\beta_1}) \cup [\gamma_1 + 2n_1 + 2, \gamma_2 + 2n_2), \end{aligned}$$

as required. (Note that $n_2 = 0$ since $\beta_2$ is a limit ordinal). $\square$

**Lemma 4.** *The map $T_\beta : U_\beta \to 2^{V_\beta} \setminus \{\emptyset\}$ is injective if $1 \leq \beta \leq \beta_0$.*

*Proof.* Suppose that $e_{\beta_1}$ and $e_{\beta_2}$ are distinct elements in $U_\beta$, with $\beta_1 < \beta_2 < \beta$. Write $\beta_2 = \gamma_2 + n_2$. It follows from Lemma 3 that $\gamma_2 + 2n_2 \in T_\beta(e_{\beta_1}) \setminus T_\beta(e_{\beta_2})$. $\square$

**Proposition 5.** *If $1 \leq \beta \leq \beta_0$, then $T_\beta(x \vee y) = T_\beta(x) \cup T_\beta(y)$ for all $x, y \in U_\beta$. In particular, $T_\beta(x) \subseteq T_\beta(y)$ if $x \leq y$,.*

*Proof.* The second statement follows easily from the first. We prove the first statement by induction on $\beta$. The result is clear if $\beta = 1$. Suppose that the assertion is true for some $\beta$, $1 \leq \beta < \beta_0$. Let $x = e_{\beta_1}, y = e_{\beta_2} \in U_{\beta+1}$. We may assume that $\beta_1 < \beta_2 < \beta + 1$. Write $\beta = \gamma + n$, and $\beta_i = \gamma_i + n_i$, $i = 1, 2$, and consider two cases.

Case 1. $\beta_1 < \beta_2 < \beta$.
By Lemma 3 and the inductive hypothesis,

$$\begin{aligned}
&T_{\beta+1}(e_{\beta_1}) \cup T_{\beta+1}(e_{\beta_2}) \\
&\qquad = T_{\beta_1+1}(e_{\beta_1}) \cup [\gamma_1 + 2n_1 + 2, \gamma + 2n) \cup \\
&\qquad\qquad \cup T_{\beta_2+1}(e_{\beta_2}) \cup [\gamma_2 + 2n_2 + 2, \gamma + 2n) \cup \{\gamma + 2n, \gamma + 2n + 1\} \\
&\qquad = T_\beta(e_{\beta_1}) \cup T_\beta(e_{\beta_2}) \cup \{\gamma + 2n, \gamma + 2n + 1\} \\
&\qquad = T_\beta(e_{\beta_1} \vee e_{\beta_2}) \cup \{\gamma + 2n, \gamma + 2n + 1\} \\
&\qquad = T_{\beta+1}(e_{\beta_1} \vee e_{\beta_2}),
\end{aligned}$$

by definition of $T_{\beta+1}$, since $e_{\beta_1} \vee e_{\beta_2} \neq e_\beta$ by part (b) of Lemma 2.

Case 2. $\beta_1 < \beta_2 = \beta$.
In this case,

$$T_{\beta+1}(x) \cup T_{\beta+1}(y) = \bigcap_{e_\beta < z \in U_\beta} [T_\beta(x) \cup T_\beta(z)] \cup \{\gamma + 2n, \gamma + 2n + 1\}.$$

Note that by part (b) of Lemma 2, $x \vee e_\beta = e_\xi$ for some $\xi \leq \beta_1$. Hence, $x \vee e_\beta \in U_{\beta+1} \setminus \{e_\beta\} = U_\beta$. Thus, it suffices to show that

$$\bigcap_{e_\beta < z \in U_\beta} [T_\beta(x) \cup T_\beta(z)] = T_\beta(x \vee e_\beta) = T_\beta(x \vee y).$$

Since $e_\beta < x \vee e_\beta \in U_\beta$, $\bigcap_{e_\beta < z \in U_\beta} T_\beta(z) \subseteq T_\beta(x \vee e_\beta)$. By the inductive hypothesis, $T_\beta(x) \subseteq T_\beta(x \vee e_\beta)$. It follows that $\bigcap_{e_\beta < z \in U_\beta} [T_\beta(x) \cup T_\beta(z)] \subseteq T_\beta(x \vee e_\beta)$. On the other hand, if $e_\beta < z \in U_\beta$, then $x \vee e_\beta \leq x \vee z \in U_\beta$. By the inductive hypothesis, $T_\beta(x \vee e_\beta) \subseteq T_\beta(x \vee z) = T_\beta(x) \cup T_\beta(z)$. Therefore, $T_\beta(x \vee e_\beta) \subseteq \bigcap_{e_\beta < z \in U_\beta} [T_\beta(x) \cup T_\beta(z)]$.

Suppose that $\beta$ is a limit ordinal and the Proposition holds for all $\beta' < \beta$. Let $x, y \in U_\beta$. We may assume that $x = e_{\beta_1}$ and $y = e_{\beta_2}$ for some $\beta_1 < \beta_2 < \beta$. Let $\beta_i = \gamma_i + n_i$, $i = 1, 2$. Using Lemma 3 and the inductive hypothesis,

$$\begin{aligned}
T_\beta(x) \cup T_\beta(y) &= T_{\beta_1+1}(e_{\beta_1}) \cup T_{\beta_2+1}(e_{\beta_2}) \cup [\gamma_1 + 2n_1 + 2, \beta) \\
&= T_{\beta_2+1}(e_{\beta_1}) \cup T_{\beta_2+1}(e_{\beta_2}) \cup [\gamma_2 + 2n_2 + 2, \beta) \\
&= T_{\beta_2+1}(e_{\beta_1} \vee e_{\beta_2}) \cup [\gamma_2 + 2n_2 + 2, \beta).
\end{aligned}$$

By (b) of Lemma 2, $e_{\beta_1} \vee e_{\beta_2} = e_\eta$ for some $\eta \leq \beta_1$. By Lemma 3,

$$\begin{aligned}
T_{\beta_2+1}(e_{\beta_1} \vee e_{\beta_2}) &= T_{\eta+1}(e_\eta) \cup [\gamma_\eta + 2n_\eta + 2, \gamma_2 + 2n_2 + 2), \\
\text{and } T_{\beta+1}(e_{\beta_1} \vee e_{\beta_2}) &= T_{\eta+1}(e_\eta) \cup [\gamma_\eta + 2n_\eta + 2, \beta),
\end{aligned}$$

where $\eta = \gamma_\eta + n_\eta$. Combining the three preceding equations gives $T_\beta(x) \cup T_\beta(y) = T_\beta(x \vee y)$. $\square$

*Proof of Theorem 1.* Since $L = U_{\beta_0}$, Theorem 1 follows immediately from Lemma 4 and Proposition 5 by taking $\beta = \beta_0$ in each instance. $\square$

## 2. Good Lorentz Functions

A *Lorentz sequence* is a non-increasing sequence $(w(n))_{n=1}^{\infty}$ of positive numbers such that $w(1) = 1$, $\lim_n w(n) = 0$ and $\sum_{n=1}^{\infty} w(n) = \infty$. A Lorentz sequence is $C$-*submultiplicative* if $S(mn) \leq CS(m)S(n)$ for all $m, n \in \mathbb{N}$, where $S(n) = \sum_{k=1}^{n} w(k)$. In [4, §2], an infinite sequence of 1-submultiplicative Lorentz sequences is constructed so that the maxima of any two incomparable *finite* subsets are incomparable (see [4, Proposition 2.6]). For our purpose, we require an infinite sequence of $C$-submultiplicative Lorentz sequences so that the supremum of any (finite or infinite) subset remains a $C$-submultiplicative Lorentz function, and that the suprema of any two incomparable (finite or infinite) subsets are incomparable (Proposition 10). This is done by tweaking the arguments in [4, §2]. Following [4], we will find it more convenient to work with functions defined on real intervals. If $2 \leq N < \infty$, a *good Lorentz function* (GLF) on $(0, N]$ is a function $w : (0, N] \to (0, \infty)$ such that

(1) $w(x) = 1$, $x \in (0, 2]$,
(2) $w$ is nonincreasing, and
(3) If $1 \leq x, y \leq xy \leq N$, then $\int_0^{xy} w \leq \int_0^x w \cdot \int_0^y w$.

A *GLF on* $(0, \infty)$ (or simply a GLF) is a function $w : (0, \infty) \to (0, \infty)$ such that $w_{|(0,N]}$ is a GLF on $(0, N]$ for any $N \geq 2$, $\lim_{x\to\infty} w(x) = 0$ and $\int_0^{\infty} w = \infty$. It is an easy exercise to verify that if $w$ is a GLF, then $(w(n))_{n=1}^{\infty}$ is a 4-submultiplicative Lorentz sequence.

If $(u_i)$ is a finite or infinite sequence of real-valued functions with pairwise disjoint domains, let $\oplus_i u_i$ denote the set theoretic union. The constant 1 function with domain $I$ is denoted by $1_I$. We now recall the relevant facts from [4]. Note that the quantity $S(x)$ there corresponds to $\int_0^x w$ in our notation.

**Lemma 6.** [4, Lemma 2.2] *Let $w$ be a GLF on $(0, N]$, $N \geq 2$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, $w \oplus \varepsilon 1_{(N,N^2]}$ is a GLF on $(0, N^2]$.*

Repeated applications of Lemma 6 yield

**Lemma 7.** *Let $G$ be a finite set of GLF's on $(0, N]$, $N \geq 2$. For any $N' > N$ and any $\varepsilon > 0$, there is a function $v : (N, N'] \to (0, \infty)$ such that $w \oplus v$ is a GLF on $(0, N']$ for all $w \in G$, $v(x) \leq \varepsilon$, $x \in (N, N']$, and $\int_N^{N'} v < \varepsilon$.*

On the other hand, the proof of [4, Lemma 2.4] allows us to obtain GLF extensions with large total weight.

**Lemma 8.** *Let $G$ be a finite set of GLF's on $(0, N]$, $N \geq 2$ and set $K = \min_{w\in G} \int_0^N w$. For any $\varepsilon > 0$, there is a function $v : (N, N'] \to (0, \infty)$, $N' > N$, such that*

(1) *For all* $w \in G$, $w \oplus v$ *is a GLF on* $(0, N']$,
(2) $v(x) \le \varepsilon$, $x \in (N, N']$,
(3) $\int_N^{N'} v \ge \frac{K}{2}$.

We may repeat the preceding lemma to obtain

**Lemma 9.** *Let* $G$ *be a finite set of GLF's on* $(0, N]$, $N \ge 2$. *For any* $K < \infty$ *and any* $\varepsilon > 0$, *there is a function* $v : (N, N'] \to (0, \infty)$, $N' > N$, *such that*
(1) *For all* $w \in G$, $w \oplus v$ *is a GLF on* $(0, N']$,
(2) $v(x) \le \varepsilon$, $x \in (N, N']$,
(3) $\int_N^{N'} v \ge K$.

**Proposition 10.** *There exists an infinite sequence* $(w_p)_{p=1}^{\infty}$ *of GLF's on* $(0, \infty)$ *such that for every nonempty* $M \subseteq \mathbb{N}$ *and every* $p' \notin M$,
(1) $w_M = \sup_{p \in M} w_p$ *is a GLF on* $(0, \infty)$,
(2)
$$\sup_n \frac{\int_0^n w_{p'}}{\int_0^n w_M} = \infty.$$

*Proof.* The desired family of incomparable GLF's is constructed by defining its elements inductively on successive intervals. On each of the segments, each of the $w_p$'s is chosen to be either "high" or "low".

Let $((p_i, q_i))_{i=1}^{\infty}$ be an enumeration of $\{(p, q) : p < q, p, q \in \mathbb{N}\}$ and fix a positive sequence $(\varepsilon_i)$ decreasing to 0. For all $p \in \mathbb{N}$, define $w_p^0 : (0, 2] \to (0, \infty)$ by $w_p^0(x) = 1$. Set $G_0 = \{w_p^0 : p \in \mathbb{N}\}$.

Assume that for some $i \in \mathbb{N}$, functions $w_p^j : (N_{j-1}, N_j] \to (0, \infty)$, $0 \le j < i$ ($N_{-1} = 0$, $N_0 = 2$), $p \in \mathbb{N}$, have been defined so that $G_{i-1} = \{w_{r_0}^0 \oplus \cdots \oplus w_{r_{i-1}}^{i-1} : r_0, \ldots, r_{i-1} \in \mathbb{N}\}$ is a finite set of GLF's on $(0, N_{i-1}]$ and that $\{w_p^j : p \in \mathbb{N}\}$ is a totally ordered set of functions (in the pointwise order) for each $j \in [0, i)$. Set $K_{i-1} = \int_0^{N_{i-1}} \max G_{i-1}$, where by $\max G_{i-1}$ we mean the pointwise maximum of the set of functions $G_{i-1}$. By Lemma 9, choose a function $w_{p_i}^i$ on $(N_{i-1}, N_i]$, $N_i > N_{i-1}$, such that $w \oplus w_{p_i}^i$ is a GLF on $(0, N_i]$ for all $w \in G_{i-1}$, that $w_{p_i}^i(x) \le \varepsilon_i$ for all $x \in (N_{i-1}, N_i]$ and that $\int_{N_{i-1}}^{N_i} w_{p_i}^i \ge q_i K_{i-1}$. On the other hand, by Lemma 7, there exists $v$ on $(N_{i-1}, N_i]$ such that $w \oplus v$ is a GLF on $(0, N_i]$ for all $w \in G_{i-1}$, that $v(x) \le w_{p_i}^i(N_i)$ for all $x \in (N_{i-1}, N_i]$ and that $\int_{N_{i-1}}^{N_i} v \le 1$. Define $w_p^i = v$ for all $p \ne p_i$. Note that $G_i = \{w \oplus w_p^i : w \in G_{i-1}, p \in \mathbb{N}\}$ is a finite set of GLF's on $(0, N_i]$. Obviously, the set $\{w_p^i : p \in \mathbb{N}\} = \{w_{p_i}^i, v\}$ is totally ordered. This completes the inductive construction. Define $w_p = \oplus_i w_p^i$, $p \in \mathbb{N}$. Observe that $K_0 = 2$ and $K_i \ge K_{i-1} + q_i K_{i-1} \ge 3K_{i-1}$. Hence $K_i \to \infty$. Thus

$$N_i \ge N_i - N_{i-1} \ge \int_{N_{i-1}}^{N_i} w_{p_i}^i \ge q_i K_{i-1} \to \infty.$$

Hence $w_p$ is defined on $(0,\infty)$ for all $p\in\mathbb{N}$. If $\emptyset\neq M\subseteq\mathbb{N}$, let $w_M=\sup_{p\in M}w_p$. We claim that $w_M$ is a GLF on $(0,\infty)$. By definition, $w_{M|(0,N_i]}\in G_i$ for all $i\in\mathbb{N}$. Thus $w_M$ is a GLF on $(0,N_i]$ for all $i\in\mathbb{N}$. Also note that $w_M(x)\leq\varepsilon_i$ for all $x\in(N_{i-1},N_i]$. Therefore, $\lim_{x\to\infty}w_M(x)=0$. Furthermore, since $w_M=w^i_{p_i}$ on $(N_{i-1},N_i]$ if $p_i\in M$,

$$\int_0^\infty w_M>\sup_{\{i:p_i\in M\}}\int_{N_{i-1}}^{N_i}w^i_{p_i}\geq\sup_{\{i:p_i\in M\}}q_iK_{i-1}.$$

Because of the enumeration, $p_i\in M$ holds for infinitely many $i$. It follows that $\int_0^\infty w_M=\infty$. This shows that $w_M$ is a GLF on $(0,\infty)$.

Finally, note that for all $i$ such that $p_i\notin M$, $\int_{N_{i-1}}^{N_i}w_M\leq 1$ by construction. In particular, if $p'\notin M$, then for all $i$ such that $p_i=p'$,

$$\begin{aligned}\int_0^{N_i}w_M&\leq\int_0^{N_{i-1}}w_M+\int_{N_{i-1}}^{N_i}w_M\\&\leq\int_0^{N_{i-1}}\max G_{i-1}+\max_{p\in M}\int_{N_{i-1}}^{N_i}w_p\leq K_{i-1}+1.\end{aligned}$$

On the other hand, for all such $i$,

$$\int_0^{N_i}w_{p'}\geq\int_{N_{i-1}}^{N_i}w_{p'}=\int_{N_{i-1}}^{N_i}w^i_{p_i}\geq q_iK_{i-1}.$$

Hence

$$\sup_n\frac{\int_0^n w_{p'}}{\int_0^n w_M}=\infty.$$

□

Given a Lorentz sequence $(w(n))_{n=1}^\infty$ and $1\leq p<\infty$, the Lorentz sequence space $d(w,p)$ consists of all real sequences $(a_n)$ such that $\sum a_n^*w_n<\infty$, where $(a_n^*)$ denotes the non-increasing rearrangement of $(|a_n|)$.

**Corollary 11.** *Let $(w_p)_{p=1}^\infty$ be as above. For every $M\subseteq\mathbb{N}$, and $p\notin M$, the unit vector basis of $d(w_M,1)$ does not dominate that of $d(w_p,1)$.*

*Proof.* Let $(v_i)$ and $(u_i)$ denote the respective unit vector bases of $d(w_p,1)$ and $d(w_M,1)$. According to Proposition 10, for any $K<\infty$, there exists $N\in\mathbb{N}$ such that $\int_0^{N+1}w_p\geq K\int_0^{N+1}w_M$. Then

$$\begin{aligned}\Big\|\sum_{i=1}^N v_i\Big\|=\sum_{i=1}^N w_p(i)&\geq\int_1^{N+1}w_p=\int_0^{N+1}w_p-1\\&\geq K\int_0^{N+1}w_M-1\geq K\sum_{i=1}^N w_M(i)-1=K\Big\|\sum_{i=1}^N u_i\Big\|-1.\end{aligned}$$

The result follows since $K$ is arbitrary. □

## 3. Countable Semilattices of Spreading Models

In this section, we show that every countable semilattice without an infinite increasing sequence is order isomorphic to some $SP_w(X)$. If $(x_i)$ and $(y_i)$ are sequences in the Banach spaces $X$ and $Y$ respectively, let $(x_i) \oplus (y_i)$ denote the sequence $(z_i) = (x_i, y_i)$ in the direct sum $X \oplus Y$. The $\ell^p$-sum of an infinite sequence $(X_j)$ of Banach spaces is denoted by $(\sum_{j=1}^{\infty} \oplus X_j)_p$. We omit the easy proof of the next lemma.

**Lemma 12.** *Let $w_1 = (w_1(n))$ and $w_2 = (w_2(n))$ be Lorentz sequences. Then $w = w_1 \vee w_2 = (w_1(n) \vee w_2(n))$ is a Lorentz sequence. Moreover, if $(u_n^1)$ and $(u_n^2)$ are the respective unit vector bases of $d(w_1, 1)$ and $d(w_2, 1)$, then $(u_n^1) \oplus (u_n^2)$ is equivalent to $(u_n)$, the unit vector basis of $d(w, 1)$.*

**Lemma 13** ([4, Lemma 3.6])**.** *Let $X = (\sum_{j=1}^{\infty} \oplus X_j)_p$, where $1 \le p < \infty$ and each $X_j$ is an infinite-dimensional Banach space, and let $(\tilde{x}_i)$ be a spreading model generated by a normalized weakly null sequence in $X$. Then there exist non-negative $(c_j)_{j=0}^{\infty}$ with $\sum_{j=0}^{\infty} c_j^p = 1$ and normalized spreading models $(\tilde{x}_i^j)_i$ in $X_j$ generated by weakly null sequences such that for all scalars $(a_i)$,*

$$\|\sum_i a_i \tilde{x}_i\| = \big[\sum_{j=1}^{\infty} c_j^p \|\sum_i a_i \tilde{x}_i^j\|^p + c_0^p \sum_i |a_i|^p\big]^{1/p}. \tag{1}$$

**Remark**. If $p = 1$, the final term on the right of equation (1) may be omitted, i.e., $c_0 = 0$. In fact, according to the proof of Lemma 13 in [4, Lemma 3.6], the spreading model $(\tilde{x}_i)$ is generated by a weakly null sequence $(x_i)$ in $X$ in such a way that $c_0 = \lim \|x_i - P_i(x_i)\|$, where $P_i(x_i) = \big(x_i^1, x_i^2, \cdots, x_i^i, 0, 0, 0, \cdots\big)$. However, since $\ell^1$ has the Schur property (weakly null sequences are norm null), it is easy to see that $\lim \|x_i - P_i(x_i)\| = 0$ for any weakly null sequence $(x_i)$ in $(\sum_{j=1}^{\infty} \oplus X_j)_1$.

The following is the crucial property of Lorentz sequence spaces that we require. It can be deduced from the arguments in [1, §4]:

**Theorem 14.** [1] *Let $w = (w(n))$ be a $C$-submultiplicative Lorentz sequence and let $(u_n)$ be the unit vector basis of $d(w, 1)$. For any $\varepsilon > 0$, every normalized block basis in $d(w, 1)$ has a subsequence $(x_n)$ such that either*

(a) *$(x_n)$ is equivalent to the unit vector basis of $\ell^1$, or*
(b) *there exists $c > 0$ such that for all $(a_n) \in c_{00}$,*

$$c\|\sum a_n u_n\| \le \|\sum a_n x_n\| \le (C + \varepsilon)\|\sum a_n u_n\|. \tag{2}$$

*In particular, if $(\tilde{x}_n)$ is a spreading model generated by a normalized weakly null sequence, then $(\tilde{x}_n)$ satisfies (2) in place of $(x_n)$.*

**Theorem 15.** *Given a countable semilattice $L$ with no infinite increasing sequence, there is a Banach space $X_L$ such that $SP_w(X_L)$ is order isomorphic to $L$.*

*Proof.* By Theorem 1, there exists a countable set $V$ and an injective map $T : L \to 2^V \setminus \{\emptyset\}$ such that $T(e \vee f) = T(e) \cup T(f)$ for all $e, f \in L$. Since $V$ is countable, by Proposition 10 (and Corollary 11), there is a family $(w_v)_{v \in V}$ of 4-submultiplicative GLF's such that for each non-empty subset $M$ of $V$, $w_M = \sup_{v \in M} w_v$ is again a (4-submultiplicative) GLF. Moreover, if $p \notin M$, the unit vector basis of $d(w_M, 1)$ does not dominate that of $d(w_p, 1)$. Set $X_L = \left(\bigoplus_{e \in L} d(w_{Te}, 1)\right)_1$. For any $e \in L$, let $(u_i^e)$ be the unit vector basis of $d(w_{Te}, 1)$. $(u_i^e)$ may be regarded in an obvious way as a normalized weakly null sequence in $X_L$ which generates a spreading model equivalent to itself. Thus $[(u_i^e)]$, the equivalence class containing $(u_i^e)$, is an element of $SP_w(X_L)$. Define a map $\Theta : L \to SP_w(X_L)$ by $\Theta e = [(u_i^e)]$. We will show that $\Theta$ is a bijection such that $\Theta e_1 \le \Theta e_2$ if and only if $e_1 \le e_2$. Hence $SP_w(X_L)$ is order isomorphic to $L$.

We first show that $\Theta$ is onto. Let $[(\tilde{x}_i)]$ be an element in $SP_w(X_L)$. By Lemma 13 and the subsequent Remark, there exist a non-negative sequence $(c_e)_{e \in L}$ with $\sum c_e = 1$ and normalized spreading models $(\tilde{x}_i^e)$ in $d(w_{Te}, 1)$ such that

$$\|\sum_i a_i \tilde{x}_i\| = \sum_{e \in L} c_e \|\sum_i a_i \tilde{x}_i^e\|. \tag{3}$$

Since each $w_{Te}$ is 4-submuliplicative, according to Theorem 14, for each $e \in L$, there exists $b_e > 0$ such that

$$b_e \|\sum_i a_i u_i^e\| \le \|\sum_i a_i \tilde{x}_i^e\| \le 5 \|\sum_i a_i u_i^e\|. \tag{4}$$

Let $I = \{e \in L : c_e > 0\}$. If $I$ is infinite, write its elements in a sequence $(e_i)_{i=1}^\infty$. Since the sequence $(\vee_{i=1}^n e_i)_{n=1}^\infty$ has no strictly increasing infinite subsequence, there is a finite subset $J$ of $I$ such that $\vee_{e \in J} e \ge e'$ for all $e' \in I$. If $I$ is finite, take $J = I$. Let $f = \vee_{e \in J} e$. We claim that $(\tilde{x}_i)$ is equivalent to $(u_i^f)$. Observe that $e \le f$ for all $e \in I$. Hence $Te \subseteq Tf$ and thus $w_{Te} \le w_{Tf}$. Therefore, $(u_i^e)$ is 1-dominated by $(u_i^f)$. By (3) and (4),

$$\begin{aligned}
\|\sum_i a_i \tilde{x}_i\| &= \sum_{e \in L} c_e \|\sum_i a_i \tilde{x}_i^e\| = \sum_{e \in I} c_e \|\sum_i a_i \tilde{x}_i^e\| \\
&\le 5 \sum_{e \in I} c_e \|\sum_i a_i u_i^e\| \le 5 \sum_{e \in I} c_e \|\sum_i a_i u_i^f\| = 5 \|\sum_i a_i u_i^f\|.
\end{aligned}$$

On the other hand, by Lemma 12, $\bigoplus_{e \in J} (u_i^e)$ is equivalent to $(u_i^f)$. Using (3) and (4) again,

$$\begin{aligned}
\|\sum a_i \tilde{x}_i\| &= \sum_{e \in I} c_e \|\sum_i a_i \tilde{x}_i^e\| \ge \sum_{e \in I} c_e b_e \|\sum_i a_i u_i^e\| \\
&\ge \sum_{e \in J} c_e b_e \|\sum_i a_i u_i^e\| \ge \min_{e \in J}\{c_e b_e\} \sum_{e \in J} \|\sum_i a_i u_i^e\| \\
&\ge K \|\sum a_i u_i^f\| \text{ for some } K > 0.
\end{aligned}$$

This shows that $(\tilde{x}_i)$ is equivalent to $(u_i^f)$. Hence $\Theta f = [(u_i^f)] = [(\tilde{x}_i)]$.

Next we show that

$$e_1 \leq e_2 \ \Leftrightarrow \Theta e_1 \leq \Theta e_2. \tag{5}$$

If $e_1 \leq e_2$, then $Te_1 \subseteq Te_2$ and hence $w_{Te_1} \leq w_{Te_2}$. It follows that $[(u_i^{e_1})] \leq [(u_i^{e_2})]$. On the other hand, if $e_1 \nleq e_2$, then $T(e_1) \nsubseteq T(e_2)$. Choose $p \in T(e_1) \smallsetminus T(e_2)$. By Corollary 11, $(u_i^{e_2})$ does not dominate $(v_i)$, the unit vector basis of $d(w_p, 1)$. But obviously $(u_i^{e_1})$ dominates $(v_i)$. Hence $[(u_i^{e_1})] \nleq [(u_i^{e_2})]$. Note that (5) also implies that $\Theta$ is injective. Hence $\Theta : L \to SP_w(X_L)$ is an order isomorphism. □

**Remark.** The example given here is non-reflexive. Given a countable semilattice $L$ without an infinite increasing sequence, the $\ell^p$ $(1 < p < \infty)$ version of the space defined above, i.e., $X_p = \left(\bigoplus_{e \in L} d(w_{Te}, p)\right)_p$, which is a reflexive space, has the property that $SP_w(X_p)$ is order isomorphic to the semilattice $\hat{L} = \{a\} \cup L$, $a > e$ for all $e \in L$. We do not know how to obtain a reflexive example for general semilattices. In fact, according to the authors of [4], it is not known if there is a reflexive space $X$ such that $SP_w(X)$ is order isomorphic to $(\{\{1,2\}, \{1\}, \{2\}\}, \subseteq)$.

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